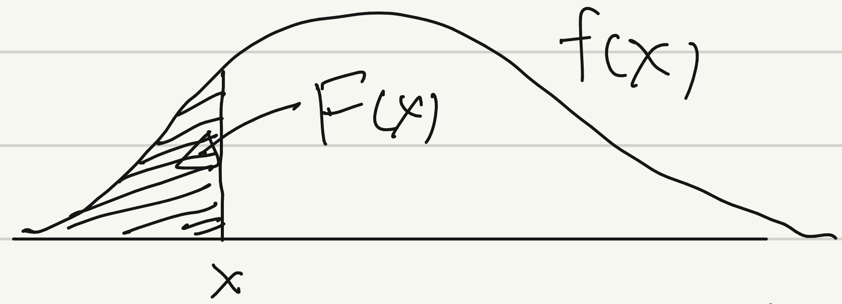
Cumulative distribution function (cdf)

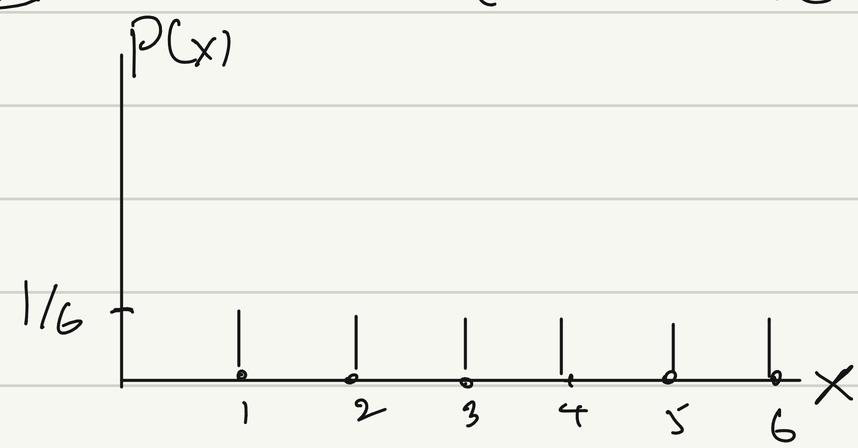
X is a random variable.
The cdf of X is

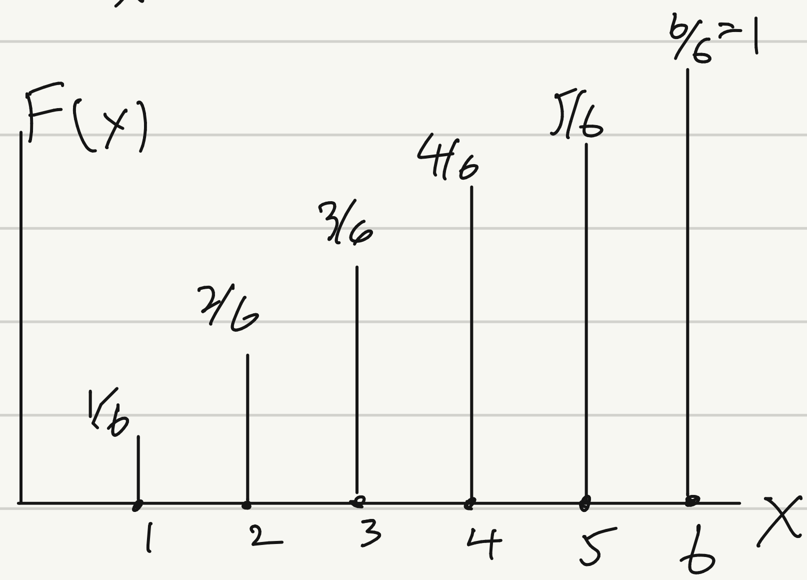
$$F(x') = P(X \le x').$$

$$\text{(continuous)} \quad = \int_{-\infty}^{x'} f(x)\,dx \quad , \; f(x) \text{ is density of } X.$$



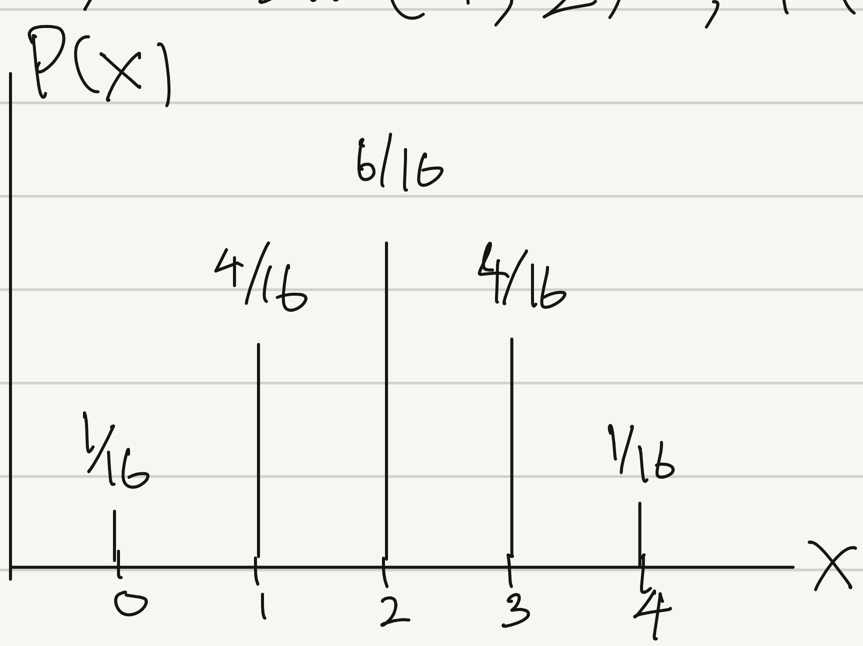
Ex X = outcome of a die.

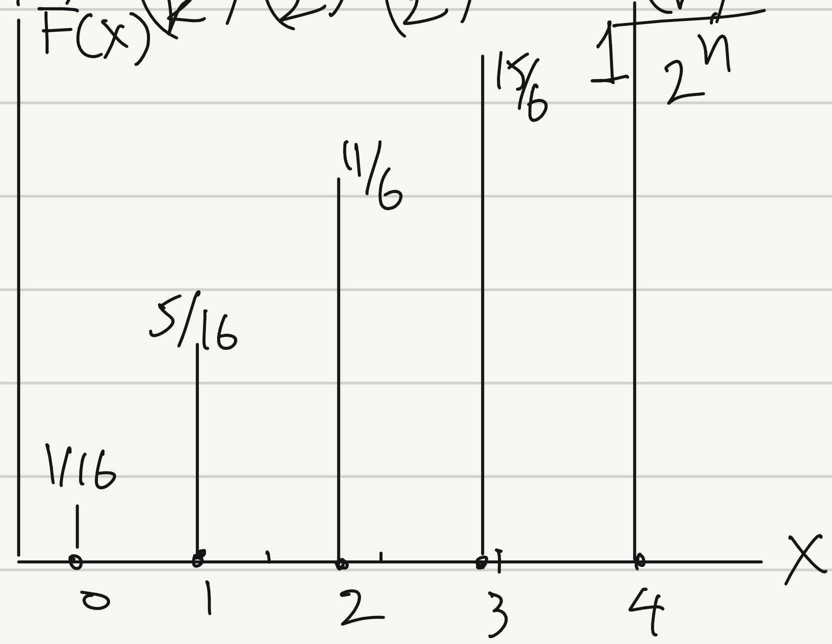




(1) F is an increasing function.
(2). F(last value of x) $= 1$

Ex $X \sim Bin(4, \frac{1}{2})$, $P(X=k) = \binom{4}{k}\left(\frac{1}{2}\right)^{k}\left(\frac{1}{2}\right)^{n-k} = \frac{\binom{4}{k}}{2^n}$



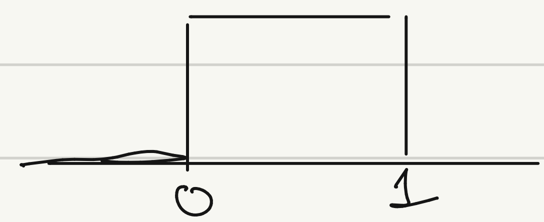


$$P(a < X \le b) = F(b) - F(a).$$

Ex: $a=1$, $b=3$, $F(3) - F(1) = P(2 \le X \le 3)$.

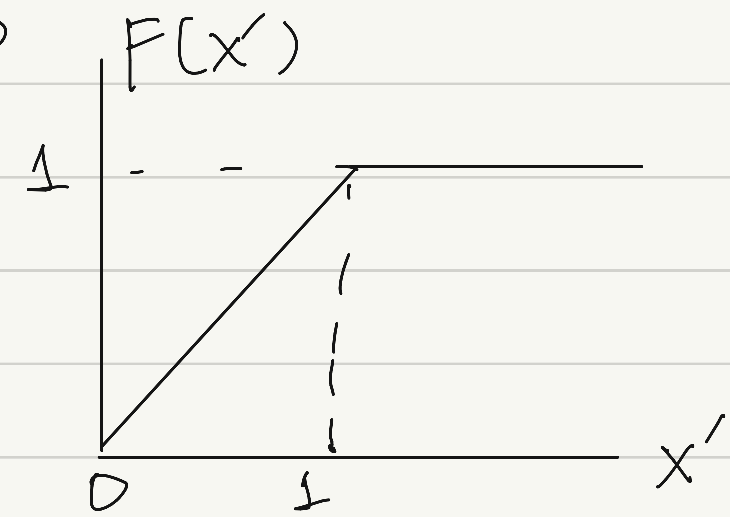
Ex: $X \sim \text{Uniform}(0,1)$.

$f(x) = 1$ if $0 \le x \le 1$ and 0 otherwise



$$F(x') = \int_{-\infty}^{x'} f(x)\,dx$$

$$= \int_0^{x'} \overset{x^0}{1}\,dx = x^1 \Big|_{x=0}^{x=x'} = x' - 0 = x'$$



Fact: X is r.v. with cdf $F(x')$

(Fundamental theorem of Calculus).

$$\frac{d}{dx'} \boxed{\underbrace{\int_0^{x'} f(x)\,dx}_{\text{function of } x'}} = f(x')$$

$$\Rightarrow \frac{d}{dx'} F(x') = f(x').$$

Ex: X is r.v. with cdf $F(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \\ 0 & \text{if } x < 0 \end{cases}$

The pdf of X is

$$f(x) = \frac{d}{dx} F(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & x > 1 \\ 0 & x < 0 \end{cases}$$

Check: $\int_0^1 f(x)\,dx = \int_0^1 2x\,dx = x^2 \Big|_{x=0}^{1} = 1.$

Two continuous r.v.

$P(x, y) \longrightarrow f(x, y)$. Joint density function of X and Y.

$P(a \le X \le b) \longrightarrow \int_a^b f(x)\,dx$

$P(a \le X \le b,\ c \le Y \le d) \longrightarrow \int_c^d \int_a^b f(x, y)\,dx\,dy$

$(a \le X \le b \text{ and } c \le Y \le d)$

Condition: $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\,dx\,dy = 1.$

Double integration

Ex. $\int_0^1\int_0^1 x^2y\,dx\,dy = \int_0^1 \int_0^1 x^2y\,dx\,dy$

$= \int_0^1 y\,\frac{x^3}{3}\Big|_{x=0}^{1}\,dy$

$= \int_0^1 y\left(\frac{1}{3}-0\right)dy.$

$= \int_0^1 \frac{y}{3}\,dy$

$= \frac{1}{3}\,\frac{y^2}{2}\Big|_{y=0}^{1}$

$= \frac{1}{3}\left(\frac{1}{2}-0\right) = \frac{1}{6}.$

Fact: (Fubini's theorem):

$$\int\int f(x,y)\,dx\,dy = \int\int f(x,y)\,dy\,dx.$$

Ex: $\int_{-1}^{1}\int_2^3 \frac{x}{\log y}\,dy\,dx = \int_2^3\int_{-1}^{1}\frac{x}{\log y}\,dx\,dy$

$= \int_2^3 \frac{1}{\log y}\,\frac{x^2}{2}\Big|_{x=-1}^{=1}\,dy.$

$= \int_2^3 \frac{1}{\log y}\left(\frac{1}{2}-\frac{1}{2}\right)dy$

$= \int_2^3 0\,dy = 0.$

Fact: $\iint f(x)\cdot \boxed{g(y)}\,dx\,dy$

$= \int g(y) \boxed{\int f(x)\,dx}\, dy$

$= \int f(x)\,dx \cdot \int g(y)\,dy.$

Ex: (2-dim uniform distribution)

$f(x,y) = C$ if $1 \le x \le 3$ and $2 \le y \le 5$
and 0 otherwise

Q: Find C that makes f a density function.

$$1 = \int_2^5\int_1^3 f(x,y)\,dx\,dy = \int_2^5\int_1^3 C\,dx\,dy$$

f(x) f(y)

$$= C\int_2^5\int_1^3 1\cdot 1\,dx\,dy$$

$$= C\int_1^3 1\,dx \int_2^5 1\,dy$$

$$= C\cdot x\Big|_{x=1}^{3}\ y\Big|_{y=2}^{5}$$

$$= C\cdot(3-1)\cdot(5-2)$$

$$= C\cdot 2\cdot 3 = 6C.$$

$$\to \quad C = \frac{1}{6}.$$

Q: $f(x,y) = \frac{1}{6}$ if $1 \le x \le 3$ and $2 \le y \le 5$

0 otherwise.

$P(1 \le X \le 2,\ 3 \le y \le 4)$

$= \int_3^4 \int_1^2 f(x,y)\,dx\,dy$

$= \int_3^4 \int_1^2 \frac{1}{6}\,dx\,dy$

$= \frac{1}{6} \int_3^4 \int_1^2 1\,dx\,dy$

$= \frac{1}{6} \int_1^2 1\,dx \int_3^4 1\,dy$

$= \frac{1}{6}(2-1)\cdot(4-3)$

$= \frac{1}{6} \cdot 1 \cdot 1 = \frac{1}{6}.$

CDF X, Y two continuous variables.

$F(x', y') = P(X \le x', Y \le y')$.

(continuous) $= \int_{-\infty}^{y'} \int_{-\infty}^{x'} f(x,y)\,dx\,dy$

Ex. $f(x,y) = \frac{1}{6}x^2y$ if $0 \le x \le 1$ and $0 \le y \le 1$
and 0 otherwise.

$$F(x',y') = \int_0^{y'}\int_0^{x'} f(x,y)\,dx\,dy$$

$f(x)$ $g(y)$

$$= \int_0^{y'}\int_0^{x'} \frac{1}{6}x^2y\,dx\,dy.$$

$$= \frac{1}{6}\int_0^{x'} x^2\,dx \int_0^{y'} y\,dy.$$

$$= \frac{1}{6} \cdot \frac{x^3}{3}\Big|_{x=0}^{x'} \cdot \frac{y^2}{2}\Big|_{y=0}^{y'}$$

$$= \frac{1}{6}\left(\frac{x'^3}{3} - 0\right)\left(\frac{y'^2}{2} - 0\right) = \frac{x'^3y'^2}{36}.$$

Fact $\dfrac{d}{dy'}\dfrac{d}{dx'}\displaystyle\int_{-\infty}^{y'}\int_{-\infty}^{x'} f(x,y)\,dx\,dy = f(x',y').$

$$\frac{d}{dx}\frac{d}{dy}F(x,y) = \frac{d}{dy}\frac{d}{dx}F(x,y) = f(x,y).$$

Ex. $F(x,y) = \begin{cases} xy^2 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1. \\ 1 & \text{if } x > 1 \\ 1 & \text{if } y > 1 \\ 0 & \text{otherwise} \end{cases}$

$$f(x,y) = \frac{d}{dy}\frac{d}{dx}F(x,y) = \frac{d}{dy}y^2 = \begin{cases} 2y. & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & x > 1 \\ 0 & y > 1 \\ 0 & \text{otherwise} \end{cases}$$

Check: $\int_0^1\int_0^1 f(x,y)\,dx\,dy$

$$= \int_0^1\int_0^1 \overbrace{1}^{f(x)}\cdot\overbrace{2y}^{g(y)}\,dx\,dy.$$

$$= \int_0^1 1\,dx \int_0^1 2y\,dy.$$

$$= (1-0)\cdot y^2\Big|_{y=0}^{1}$$

$$= 1\cdot(1-0)$$

$$= 1.$$